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Solution of problems of oblique penetration of axisymmetric projectiles into soft soil based on local interaction models☆

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ABSTRACT

A comparative analysis of the solutions of the three-dimensional problem of the oblique penetration of a rigid body into soft soil is carried out arsing interaction models based on one-dimensional solutions of the problem of the spherical cavity expansion. Both the well-known self-similar analytical solutions for an incompressible medium as well as the generalized solution for a compressible elastoplastic medium with separation of the shock wave which arises are considered. Use of the incompressible medium hypothesis, disregarding flow separation, in estimating the maximum values of the resistive forces leads to large errors. Taking account of compressibility enables the resistive forces to be refined appreciably and enables a satisfactory estimate of the deviation of the trajectories of bodies from the initial direction of motion to the obtained. In the proposed method of solving oblique penetration problems, a three-dimensional problem is reduced, on the basis of the plane sections hypothesis and disregarding peripheral mass and momentum flows, to the combined solution of a number of axisymmetric problems for each meridional section. It is shown that, with well-known local interaction models, this approach enables the reliability of the calculation of both the force and the kinematic characteristics of the penetration process to be increased considerably due to the fact that the dynamics of the free surface and cavitation effects of the covitating flow are taken into account.

Investigations[1–13] into the impact and penetration of rigid bodies into porous media are widely used, for example, in searching for the optimal shapes of a penetrating body[14–18]. An analysis of the experimental data on the penetration of a rigid projectile into sandy and argillaceous[9–11] soils, porous aluminium[12], limestone[13] and other media reveals their instability which justifies the development of simplified approaches to the modelling of transient penetration processes in geomaterials, among which analytical and numerical-analytical methods can be distinguished.

Approximate solutions, reflecting the main features of the penetration of a blunt, absolutely rigid body into an elastoplastic medium in the case of normal[19,20] and oblique[21] impact, have been obtained earlier by separating of the system of equations of the one-dimensional motion along the axis of penetration. The entry of a rigid sphere at an oblique angle into a half space occupied by an elastoplastic medium, the shear properties of which were specified in accordance with the model in Ref.[23], has been analysed.[22] Expressions for estimating the normal and shear stresses on the surface of the sphere have been obtained[22] by solving the one-dimensional, self-similar, oblique impact problem and the volume deformation of the medium was matched for each impact velocity in accordance with the experimental data.[24] A formulation and investigation of a self-similar problem on the penetration of a rigid body along a normal into soil (and the pentration into an inviscid barotropic gas from within it) similarity and dimension theory[26] have been presented.[25] The analytical solutions of problems on the penetration of a cone of finite aperture angle into a plastic gas are known,[2] with following assumptions: a) the particles of the medium move along a normal to the projectile surface in a domain bounded by the cone surface and the attached shock wave (SW), b) the soil only changes its density in the SW, and the medium behind the SW is assumed to be incompressible. The solution for the initial stage of the impact of a blunt body on a soil half space has been obtained under the same assumptions[27]. The well known[28] solutions of problems on the penetration of a thin body into elastoplastic media have also been applied[29] to soil, since the assumption that the body is thin enables one to reduce the three-dimensional problem to a one-dimensional problem. Problems of the-incompressible elastoplastic

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flow past a blunt rigid body have been investigated[29], using the Saint Venant–Mises plastic model, by the small parameter method and estimates of the dimensions of the plastic zone, similar to the shape of the body past which the flow occurs, and the strength corrections to the normal stress and the resistive force acting on the body due to the flow are presented. The special features of the motion of extremely thin bodies in an elastoplastic medium with flow separation have been studied[30] theoretically and experimentally. An asymptotic analysis of the solution of the problem of cavitating flow past a body has been carried out on the assumption that the flow is completely determined by shear strength and that the compressibility is unimportant. A qualitative investigation of the stability of the motion of thin rigid bodies of revolution in an elastoplastic medium has been carried out[31] under the same assumptions using the solutions of model flow problems[29] and proposed separation criteria. The self-similar solution of the problem of the flow of a rigidly plastic incompressible Mises–Schleicher medium past an infinite cone has been obtained[32] and its use in determining the yield stress has been demonstrated. The symmetric flow of a supersonic porous medium past a cone under conditions with an attached SW has been considered[33]; the medium was simulated by a non-linear compressible fluid with an adiabatic shock in the form of a known linear relation between the SW velocity and the mass velocity behind the SW front.

A number of papers have been concerned with developing and investigating numerical-analytical methods for calculating the parameters of the interaction between a projectile and the medium being penetrated, which encounter serious difficulties in the numerical solution of similar problems in a full three-dimensional formulation on the basis of the soil moduli.[1,23]

Techniques[34,35] have been developed for approximately solving problems of the oblique impact of a rigid body on a plate made of an incompressible, ideally plastic material. The problem of the oblique penetration of a sharp cone into soil has been solved numerically[36] using previously obtained results.[2] A method has been proposed[37] for determining the motion and the finite depth of penetration of a rigid, axi symmetric projectile into a compressible porous medium (clay or shale). The medium was assumed to be elasto-ideally plastic; the magnitudes of the deformations and displacements of the soil in the residual deformation zone which is separated out were determined by the change in the kinetic energy of the projectile in the step of the calculations. The method was developed[38] for soft soil characterized by a non-linear compressibility diagram and a yield stress which depends linearly on the pressure.

An approximate approach in which the pressure at each point of the lateral surface of the projectile is identified with the pressure on the inner surface of a spherical cavity, expanding in the unbounded medium from zero radius, that is, the local interaction model (LIM), has been widely used.[1,3–5,39–41] The one-dimensional motions of soil with spherical waves has been considered earlier[1,42–46] in problems of an explosion in friable soil taking account of dry friction. The solutions of a problem on the expansion of a spherical cavity in an incompressible elasto-ideally plastic medium[1–5,12,34,35,47,48] based on the LIM have been used to estimate the resistive forces and the depths of penetration of rigid and deformable projectile into soil, metal and concrete along a normal and at an oblique angle to the free surface. Modifications of the LIM are known which are based on the use of the exact solution of the dynamic problem of the expansion of a cavity in an incompressible elastoplastic medium with a Mohr–Coulomb[1,4] or a Mises–Schleicher[13] plasticity condition. Further development of this method was carried out in order to calculate the penetration of complex shaped projectiles[49,50] using associated or unassociated flow laws.[51] The parameters of the penetration process were determined[51–54] taking account of the compressibility of the medium and the negligible effect of compressibility in the case of metals at impact velocities up to 1 km/s was pointed out.[53,54]

Hence, the majority of theoretical results referring to the problem of the dynamic interaction of a rigid body with a soil medium have been obtained using the hypothesis of incompressibility. It was shown earlier[22,24] that, at the final stage of penetration at an oblique angle to the free surface, when the deviation of the trajectory of the projectile from the initial direction is formed, it is particularly important to take account of the compressibility of a porous medium and the error when using the incompressibility hypothesis will be greatest. The duration of the initial stage is determined by the time of imbedding of a projectile to a depth of the order of its diameter. The subsequent motion of projectile is rectilinear.[24]

1. Formulation of the problem

The impact and the initial stage of the plane-parallel motion of an axisymmetric body at an angle θ to the free surface of a half plane occupied by soil is considered. The *Oz* axis of the rectangular Cartesian system of coordinates is directed along the velocity V_0 of motion of the body and forms with the *Ox* axis, the plane of symmetry *xOz*, while the *Oy* axis is directed perpendicular to the *xOz* plane (Fig. 1, *a*).

The equations of motion of the penetrating body have the form

$$m\dot{V}_z = F_z, \quad m\dot{V}_x = F_x, \quad J\ddot{\omega} = M$$

where F_z, F_x and V_z, V_x are components of the principal force vector and the velocity vector of the body, M is the moment of the forces, $\ddot{\omega}$ is the angular acceleration and m and J are the mass and moment of inertia of the body.

The solution of the non-axisymmetric problem in accordance with the well known algorithm[55] reduces to the combined solution of a number of axisymmetric problems, the techniques for the solution of which are fairly well developed at the present time.[4,5,56,57]

The axisymmetric problems (Fig. 1, *b*) in a cylindrical system of coordinates *rOx* (*Oz* is the axis of symmetry) are formulated by splitting the projectile and the half space of the soil medium by the planes $zO\varphi_i$ (Fig. 1, *a*), where $tg\theta_i = tg\theta\cos\varphi_i (i = 0, 1, \ldots, N)$. The number of sections $N+1$ is determined by the angle of inclination of the velocity vector of the body to the free surface of the medium and the required accuracy of the solution of the problem.

The components of the principal force vector at each time step in the calculation are determined by summing the forces over the sections. In the case of a uniform subdivision with respect to the angular coordinate, we will have

$$F_z(t) = \left(F_z^0 + 2\sum_{i=1}^{N-1} F_z^i + F_z^N\right)/(2N), \quad F_x(t) = \left(F_r^0 + 2\sum_{i=1}^{N-1} F_r^i \cos\varphi_i - F_r^N\right)/(2N)$$

where $F_r^i, F_z^i (i = 0, 1, \ldots, N)$ are the contributions of the components of the forces resisting the penetration of the projectile into the soil obtained by solving the $N+1$ axisymmetric problems.

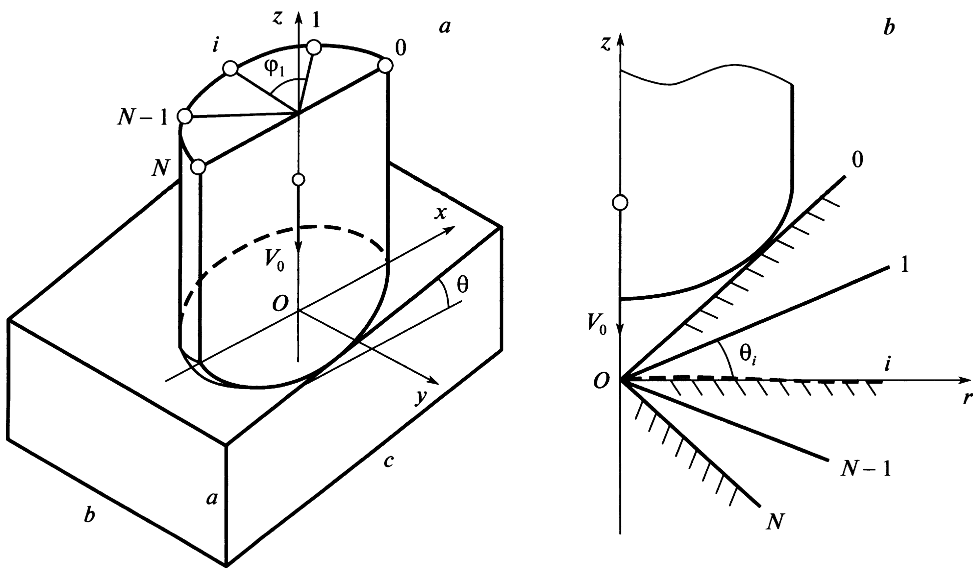

Fig. 1.

In this paper, the components of the principal force vector and the velocity vector of the body are calculated independently of one another which is justified when the axial component of the velocity vector is considerably greater than the lateral component which arises in the case of asymmetric penetration. A similar assumption is based on the results of investigations into the phenomenon of the ricochet of blunt bodies when they penetrate into sands and plasticine.[24] The fairly small deviation of the trajectory of motion of compact bodies (a small sphere, cone or cylinder) from the initial direction, which is practically independent of the entry angle of the body and decreases as the speed of impact increases, has been established experimentally. The mutual effect of the components of the principal velocity vector appears to a greater extent in the penetration of elongated bodies into soil, which is accompanied by the occurrence of rotational motion. Within the limits of the proposed technique, these processes, as well as the entry of a body at a certain small angle of attack, can be approximately calculated by the small perturbation method in a similar manner to the calculations carried out earlier.[36]

The relations of Grigoryan's mathematical model[23] of a soil are represented in the cylindrical system of coordinates *rOz* in the form of the system of differential equations

$$\frac{d\rho}{dt}+\rho(\upsilon_{r,r}+\upsilon_{z,z})=-\frac{\rho\upsilon_r}{r},\ \ \rho\frac{d\upsilon_r}{dt}-\sigma_{rr,r}-\sigma_{rz,z}=\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r},\ \ \rho\frac{d\upsilon_z}{dt}-\sigma_{rz,r}-\sigma_{zz,z}=\frac{\sigma_{rz}}{r}$$

$$D_J s_{ij}+\lambda s_{ij}=2Ge_{ij}\quad (i,j=r,z) \tag{1.1}$$

and the final relations

$$p=f_1(\rho)H(\rho_0-\rho),\quad s_{ij}s^{ij}\le\frac{2}{3}f_2^2(p) \tag{1.2}$$

where *d/dt* is the total derivative with respect to time *t*, ρ_0 and ρ are the initial and current densities. υ_r, υ_θ, υ_z are the components of the velocity vector, σ is the Cauchy stress tensor, *s* and *e* are the deviators of the stress tensor and the strain rate tensor, *H* is the Heaviside function, *p* is the pressure, D_J is a Jaumann derivative, σ_T is the yield stress, *G* is the shear modulus, and $f_1(\rho)$ *and* $f_2(\rho)$ are specified functions which characterize the bulk compressibility and shear strength of the soil. The parameter $\lambda = 0$ in the case of elastic deformation and $\lambda > 0$ if plasticity condition (1.2) is satisfied. A symbol after a comma denotes differentiation with respect to the corresponding variable and summation is carried out over repeated indices.

Three-dimensional problems are solved by the finite element method[58] and axisymmetric problems are solved using Godunov's method of the first order of accuracy.[57] The "impermeability" boundary conditions

$$\dot{u}'_\alpha=\dot{u}''_\alpha,\quad q'_\alpha=-q''_\alpha;\quad \alpha=r,z,$$

are set up where $\dot{u}_\alpha$, q_α are the components of the rate of displacement vector and the contact pressure vector. The "free surface" foundary conditions:

$$q_\alpha=0,\quad \alpha=s,\xi$$

where *s* and ξ are the directions of the tangent and normal in the local coordinate basis. The "impermeability" contact algorithm along a normal with "slip along a tangent with dry friction" is formulated in accordance with the friction law[59]

$$\dot{u}'_s=\dot{u}''_s, q'_s=-q''_s,\quad q_s=q'_s=\begin{cases}q_s, & |q_s|\le k|q_\xi|\\ k|q_\xi|\operatorname{sign}(q_s), & |q_s|>k|q_\xi|\end{cases}$$

where *k* is the coefficient of sliding friction.

The conditions on the surfaces of contact between the rigid body and the soil also take account of the phenomena of separation, the formation of free (cavitational) surfaces and the possibility of their collapse. They are formulated as a combination of impermeability on the segments of the contact surfaces at a given instant and the conditions on the free boundaries for the remaining segments. The inequality $q < q_0$, where q is the contact pressure and q_0 is a certain constant characterizing the resistance to separation, serves as the criterion for the transition from impermeability conditions to the condition on a free boundary (separation). The geometric intersection of the free surfaces of the bodies is the criterion for them to come into contact. The numerical implementation[57,58] of the contact conditions is based on the separation and tracking of the contacting and free surfaces during the calculations.

2. The problem of the spherical cavity expansion

The solution of the problem of the expansion of a spherical cavity in an elastoplastic medium in accordance with the LIM is used for the approximate estimation of the stresses acting on the projectile surface. The expansion of a cavity from a point in an unbounded medium is considered. The boundary of the cavity and the interface between the plastic and elastic zones move at a velocity V and c respectively. The solution of this one-dimensional problem is constructed in the plastic flow domain bounded by the radii $r = Vt$ and $r = ct$ and in the elastic domain adjacent to the domain of the undisturbed medium. Unlike the solutions obtained earlier,[4,5,12,47–51] in this paper the well known solution of the problem of the expansion of a cavity in a compressible medium[12] is supplemented with the conditions on the shock wave which arises when the propagation velocities of an elastic wave are greater than the magnitude of c. This considerably extends the region of applicability of the local interaction method (LIM) constructed using the solution obtained for high impact velocities.

The continuity equation and the equation for the change in momentum follow from Eqs. (1.1) and are written in Eulerian variables (spherical symmetry and a one-dimensional formulation) in the form

$$\rho\left(\upsilon_{r,r} + 2\frac{\upsilon_r}{r}\right) = -\frac{d\rho}{dt}, \quad \sigma_{rr,r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\rho\frac{d\upsilon_r}{dt}$$

The components of the Cauchy stress tensor (assumed positive in the case of compression) in the elastic region are related by Hooke's law and, in the plastic flow region, by the Mises–Schleicher plasticity condition

$$\sigma_{rr} - \sigma_{\theta\theta} = f_2(p)$$

A velocity V is specified[5] on the boundary of the cavity, which is expanding from a zero radius R_0, the external surface of the spherical layer R_∞ is stress free, conditions for the continuity of the velocities and stresses are satisfied in the unknown elastoplastic boundary and, at the initial instant, the cavity radius is equal to zero:

$$\upsilon_r\big|_{r=R_0} = V, \quad \sigma_{rr}\big|_{r=R_\infty} = 0, \quad [\upsilon_r] = [\sigma_{rr}] = [\sigma_{\theta\theta}] = 0, \quad R_0\big|_{t=0} = 0$$

The medium is described by Grigoryan's model[23] and, for simplicity, the functions $f_1(\rho)$ and $f_2(\rho)$ are assumed to be linear:

$$f_1 \equiv K(\rho/\rho_0 - 1), \quad f_2 \equiv Y + \mu p \tag{2.1}$$

where K, Y and μ are constants.

The self-similar solution of a system in the variable $\xi = r/(ct)$ in dimensionless form

$$S = \frac{\sigma_{rr}}{K_1}, \quad P = \frac{p}{K_1}, \quad U = \frac{\upsilon_r}{c}, \quad \varepsilon = \frac{V}{c}, \quad b = \frac{\rho_0}{\rho}, \quad J = \frac{\rho_0 V^2}{Y},$$

$$T = \frac{Y}{K_1}; \quad K_1 \equiv \left(1 + \frac{2}{3}\mu\right) K \equiv \rho_0 a^2$$

is next considered. A system of ordinary differential equations in S and U is obtained (a prime denotes differentiation with respect to ξ)

$$S' + 2\frac{T + \mu P}{\xi} = \frac{\beta^2}{b}(\xi - U)U', \quad U' + 2\frac{U}{\xi} = b(\xi - U)S' \tag{2.2}$$

where

$$\beta^2 = \frac{c^2}{a^2}, \quad P = \left(S - \frac{2}{3}T\right)\left(1 + \frac{2}{3}\mu\right)^{-1}, \quad \frac{\rho}{\rho_0} = S - \frac{2}{3}T + 1$$

The condition $U| = \varepsilon$, is satisfied on the boundary of the cavity $\xi = \varepsilon$ which moves with velocity V, and the elastic solution[12,48] is used at the interface between the elastic and plastic zones $\xi = 1$, the velocity c of the motion of which is unknown and is determined during the course of the solution.

In the case of supersonic motion, when $\beta = c/a > 1$ at the point $\xi = 1$, discontinuity relations are used, written for this problem in the form[60]

$$U|_{\xi=1} = 1 - \beta^{-2}, \quad S|_{\xi=1} = \left(1 + \frac{2}{3}\mu\right)P + \frac{2}{3}T, \quad \rho = \rho_0\beta^2$$

The boundary value problem for non-linear system (2.2) was solved by the shooting method in which the unknown velocity c was determined[12] iteratively until the boundary condition $|U - \varepsilon| < \delta$ had been satisfied with a specified accuracy δ. At each iteration, the

Cauchy problem was numerically solved when ξ changed from 1 to ε, that is, from the boundary of the elastoplastic interface to the cavity boundary. The Runge–Kutta method of fourth order accuracy was used.

3. Results of a comparative analysis

The results of an analysis of the applicability of the models described above to the calculation of the parameters for the penetration of a rigid sphere of radius R with a density of 7.8 g/cm^3 into soil over a range of impact velocities from 150 to 300 m/s[24] will now be presented. The constants of the model of the medium (2.1), K, G and Y were taken to be equal to 320, 160 and 0.5 MPa respectively, $\rho_0 = 2$ g/cm^3, $\mu = 1$ and $k = 0$. The three-dimensional zone, occupied by the soil is a rectangular parallelepiped (Fig. 1, a) with dimensions of the edges satisfying the equalities

$$a/6 = b/5 = c/10 = R$$

The finite-element mesh in the three-dimensional calculations[58] consisted of 30 × 150 × 75 elements. A finer difference mesh with a side of each square cell equal to $R/100$ is used in the axisymmetric calculations.[57]

The axial components of the resistance to the penetration of a sphere with an initial velocity $V_0 = 150$ m/s along a normal to the free surface are shown in Fig. 2 as a function of the dimensionless time. The values of the force F and the current time t are referred to the quantities $F_0 = \rho_0 V_0^2 \pi R^2/2$ and H/V_0 respectively. The quantity H is determined from the formula[22]

$$H/R = 1/\cos\theta + \mathrm{tg}\theta$$

The points *1* and curve *2* in Fig. 2 correspond to the results of calculations in the three = dimensional and axisymmetric formulations. The good agreement between the results of the calculations enables us to use the chosen difference mesh in axisymmetric calculations and in solving oblique penetration problems. Solutions are shown using LIMs constructed from the one-dimensional, self-similar solution (2.2) of the problem of the expansion of a spherical cavity in a compressible medium with constitutive equation (2.1) (curve *3*), of an incompressible, ideally plastic medium (curve *4*) and an incompressible medium taking account of the dependence of the yield stress on the pressure (curve *5*).

Curve *3* in Fig. 2 is close to the exact solution at the initial stage of penetration. Note that, in LIM, the maximum wetted surface of a sphere when $\theta = 0$ corresponds to the surface of the half-sphere and is attained at a depth of $H = R$. A separation of the flow is observed experimentally when $H \approx R/2$, which is also confirmed by calculations in the full formulation. Hence, for better agreement with experiments, the LIM for blunt bodies must be supplemented with a criterion for the separation of the flow of a elastoplastic medium.[31]

The components F_z/F_0 and F_x/F_0 of the principal vector of the dimensionless resistance to the penetration of a sphere into soil are shown as a function of time in Fig. 3 for the case of oblique impact (the angle $\theta = \pi/6$) with an initial velocity of 150 and 300 m/s. Good agreement is observed between the results of calculations using the proposed algorithm (curves *2*) and the results of three-dimensional calculations (points *1*) up to the instant of complete imbedding of the half-sphere into the soil. The subsequent motion of the sphere is rectilinear and is solely determined by the axial component F_z of the resistance since the component F_x vanishes. As in the case of normal impact (Fig. 2), LIM on the basis of the exact solution of system (2.2), taking account of the properties of the medium (2.1) (curves *3* in Fig. 3) give solutions

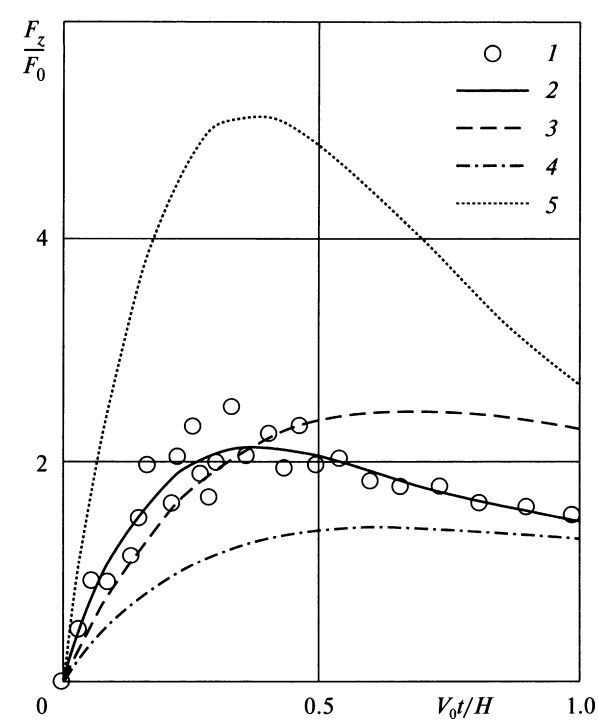

Fig. 2.

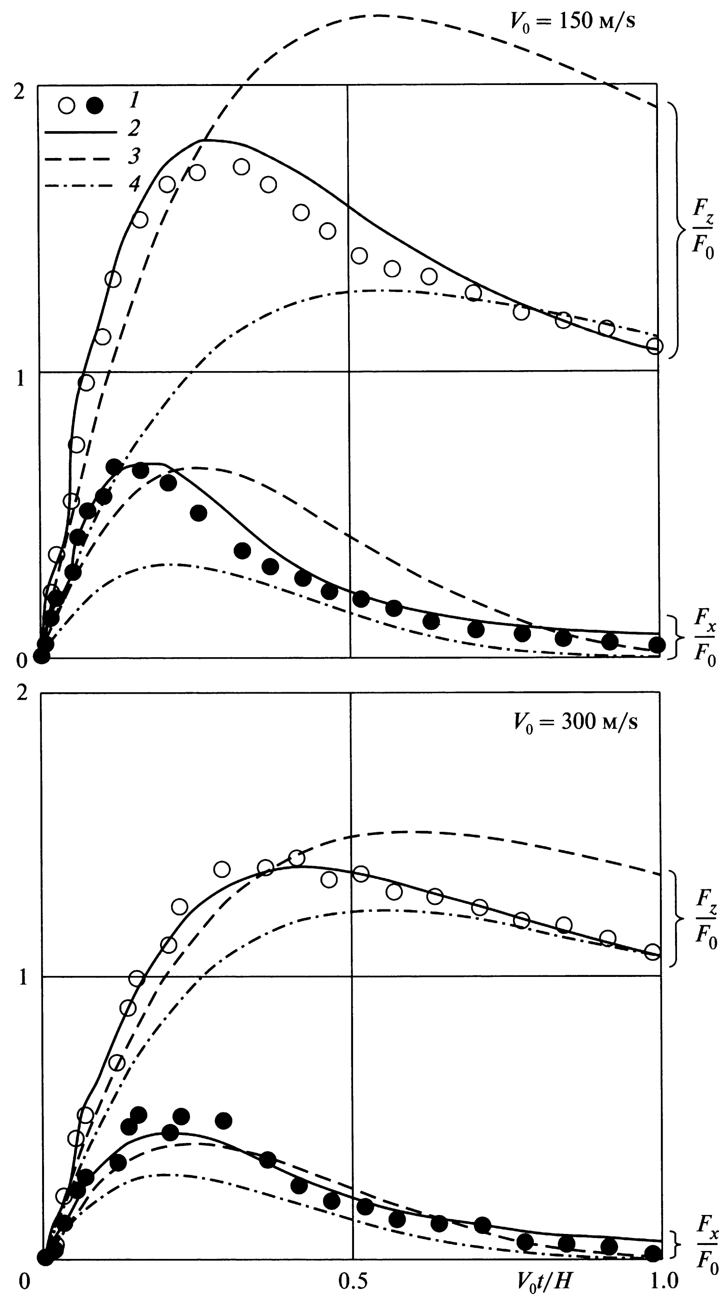

Fig. 3.

which are only close to the exact solution at the initial stage of penetration. Use of the incompressibility hypothesis in the case of a constant yield stress leads to a reduction in the resistance at the initial stage of the impact (curves *4*).

Graphs of the change in the quantity

$$\eta = \sqrt{1+\left(\frac{V_x}{V_z}\right)^2}\left(1+\mathrm{ctg}\theta_1\frac{V_x}{V_z}\right)^{-1}$$

which characterizes the change in the trajectory of the body motion as a function of the dimensionless time in the case of oblique impact ($\theta=\pi/6$) with initial velocities of 150 and 300 m/s, are shown in Fig. 4. When the body is completely imbedded in the soil, the quantity η takes a value equal to the value of the refraction coefficient, which is defined[22,24] as the ratio of the sine of the angle of approach θ_1 to the sine of the angle of departure θ_2 of a solid body from the surface of soft soil. The notation for the curves in Fig. 4 is the same as that in Fig. 2. Good agreement is observed between the results obtained using the proposed algorithm and the solution in the full formulation, as well as when using LIM on the basis of the exact solution of system (2.2), taking account of the properties of the medium: the linear compressibility and the dependence of the yield stress on the pressure.

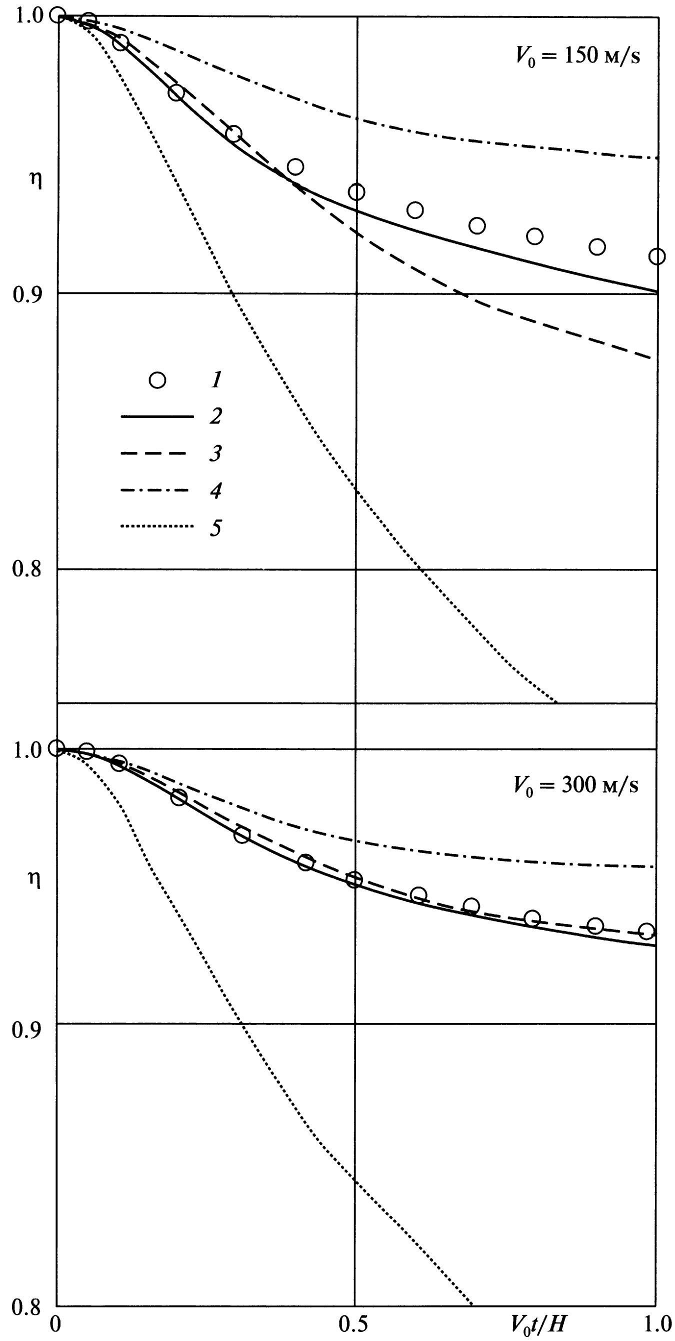

Fig. 4.

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